

ON TRAJECTORIES OF VORTICES IN THE COMPRESSIBLE FLUID ON A TWO-DIMENSIONAL MANIFOLD *

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Abstract. For the model of a compressible barotropic fluid on a two dimensional rotating Riemannian manifold we discuss a special class of smooth solutions having a form of a steady non-singular vortex moving with a bearing field. The model can be obtained from the system of primitive equations governing the motion of air over the Earth surface after averaging over the height and therefore the solution obtained can be interpreted as a tropical cyclone which is known as a long time existing stable vortex. We consider approximations of l - plane and β - plane used in geophysics for modeling of middle scale processes and equations on the whole sphere as well. We show that the solutions of the mentioned form satisfy the equations of the model either exactly or with a discrepancy which is small in a neighborhood of the trajectory of the center of vortex. We perform a numeric study of the change of the shape of the vortex affected by the neglecting the discrepancy term.

Key words. mathematical model of atmosphere, exact solution, vortex trajectory, tropical cyclone

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1. Introduction. A lot of physical models can be reduced to the equations of motion of compressible medium on a two-dimensional manifold. One of the most important is the model of dynamics of the atmosphere. The vertical scale of the atmosphere is much more larger than the horizontal one, therefore one can reduce the initial primitive three-dimensional system of equations to a two-dimensional one, convenient for describing many kinds of motions of middle and global scale, using a special procedure of averaging over the height.

There are a lot of papers where the equations governing the atmospheric motion are reduced to the two-dimensional system of incompressible Navier-Stokes equations with viscosity (see [31],[12], and references therein), on this way a lot of results on existence and uniqueness of solution to the Cauchy problem were obtained. The effect of compressibility makes the situation more difficult and the results on the global in time correctness for the viscous case are very scared (a recent state of art can be found in [14]). If we neglect the viscosity as they usually do in the meteorology (thus we get the quasilinear hyperbolic system for compressible fluid), we find ourself in a more complicated situation since the solution to the Cauchy problem basically loses its initial smoothness with time. Recent developments in the study of the Cauchy problem for the equations for compressible fluids are reviewed in [8].

Nevertheless, it is well known that there exist very stable vortices in the atmosphere that move without changing their shape for days and even weeks. Most

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notorious example is the tropical typhoon, one of the most devastating weather phenomena in the world. In practice, the most important problem is to describe (or better predict) the path of typhoon, whereas the processes inside the vortex are of theoretical interest.

In our earlier papers [29], [30] we developed a theory on possible trajectories of a stable vortex with a linear profile of velocity governed by the system of equations of compressible fluid on a plane (see Sec.4.1 for details). These vortices can be interpreted as tropical cyclones in the conservative phase of its dynamics, though they have a number of nonrealistic features. First, the components of the velocity and pressure rise unboundedly as the distance from the center increases (we call these vortices non-localized). Second, the curvature of the Earth surface was not taken into account even in a simplest form. Nevertheless, as it was shown in [30] by comparing with observational data, the theoretical trajectories of vortices can imitate the real tropical cyclone paths. In fact, they are a superposition of two circular motions and correspond to the natural circular (or rather parabolic, taking into account the change of the Coriolis parameter with the latitude) trajectories, loops, reversal points, etc. Moreover, the explicit expression for the trajectory was obtained.

In the present paper we show that the trajectories of localized vortices that model the physical process more adequately are very similar to the trajectories of the non-localized vortex with linear profile of velocity considered in [30] under some realistic assumptions on the background field of pressure.

It is worth mentioning that the exact form of equations describing the atmosphere is not known. More exactly, in the primitive system of equations we can take into account some additional forces or sources of energy, e.g. due to phase transitions, and the exact form of these terms can hardly be found, particularly after the averaging procedure.

We will base on the assumption on existence of special form of solution for the system of equations of the atmosphere dynamics (steady vortex moving in a bearing field) together with the assumption on a smallness of the source term that guarantees the existence of this special solution, at least in a domain of space that we are interested to control.

We can also look at the problem from another side and call a vector-function the " δ -approximate" solution to a certain system in a domain $D \in \mathbb{R}^2 \times \mathbb{R}_+$ if the discrepancy term Q arising after substitution of this function to the system is less than δ in the uniform norm. Thus, we can consider the term Q either as some source term that guarantees the existence of the *exact* vortex solution or as a *discrepancy*, in the latter case the vortex solution can be considered as an approximative one.

The paper is organized as follows. In Section 2, we recall the derivation of a bidimensional model of the atmosphere dynamics based on the primitive equations for compressible viscous heat-conductive gas in the physical 3D space. In Section 3, we develop a general approach to the solution in the form of "frozen" vortex moving in the exterior field. In Section 4 we demonstrate this method for the vortex solutions in the l -plane and the β -plane models and consider different shape of possible vortices (both localized and non-localized). Moreover, we compare the trajectories obtained here with the trajectory of vortex with linear velocity profile where the exact analytical result is available. Further, we perform direct numerical computation of the moving vortex in the 2D models of the l -plane and β -plane and compare the position of the center of vortex with the result obtained by our method. In Section 5 we consider the case of a sphere. We construct global exact solutions for non-rotating

sphere and study the trajectories of approximative vortices for the rotating case as well. Finally, a discussion about our topics is provided in Section 6.

2. Bidimensional models of the atmosphere dynamics. We do not dwell at the procedure of averaging and refer to our paper [30], where we use the approach by [24] and [2]. The initial (primitive) three-dimensional system relates to the motion of compressible rotating, viscous, heat-conductive, Newtonian polytropic gas [17], the resulting two-dimensional system consists of three equations for density $\varrho(t, x)$, velocity $\mathbf{U}(t, x)$ and pressure $P(t, x)$ (the equation for the velocity is vectorial):

$$\varrho(\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{U} + lL\mathbf{U}) + \nabla P = F_1, \quad (2.1)$$

$$\partial_t \varrho + \nabla \cdot (\varrho \mathbf{U}) = 0, \quad (2.2)$$

$$\partial_t P + (\mathbf{U} \cdot \nabla) P + \gamma P \operatorname{div} \mathbf{U} = F_2, \quad (2.3)$$

where F_1 and F_2 are some source terms (see [30]) for details, $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, l if the Coriolis parameter, a smooth function of x , its concrete form we discuss below. In fact, system (2.1) – (2.3) consists of equations of balance of mass, momentum and total energy, $t \in \mathbb{R}_+$, $x \in \mathcal{M}$, where \mathcal{M} is a 2D Riemannian manifold.

We can consider the system at any 2D Riemannian manifold \mathcal{M} (with or without boundary). Thus, we have to take into account the curvilinear metrics of the space and write the system in the respective curvilinear coordinates. In this case we mean by partial derivatives the covariant derivatives [37]. In particular, to model the motion of planetary scale we should use the spherical coordinates ([25]). Nevertheless, for the phenomena of middle scale it is convenient to use the approximation of plane, without taking into account the effects of curvature.

Now we recall shortly the procedure of averaging. Let $\rho, u = (u_1, u_2, u_3)$, p , denote in the *three-dimensional* density, velocity and pressure. Namely, all these functions depend on (t, x, z) , $z \in \mathbb{R}_+$. Let us introduce $\hat{\phi}$ and \bar{f} to represent for taking the average of ϕ and f over the height, respectively. The averaged values are introduced as follows: $\hat{\phi} := \int_0^\infty \phi dz$, $\bar{f} := \frac{1}{\hat{\rho}} \int_0^\infty \rho f dz$, where ϕ and f are arbitrary functions, and denote $\varrho(t, x) = \hat{\rho}$, $P(t, x) = \hat{p}$, $\mathbf{U}(t, x) = (\bar{u}_1, \bar{u}_2)$. Moreover, the usual adiabatic exponent, $\tilde{\gamma}$, is related to the “two-dimensional” adiabatic exponent γ as follows: $\gamma = \frac{2\tilde{\gamma} - 1}{\tilde{\gamma}} < \tilde{\gamma}$.

The impenetrability conditions are included in the model. These conditions ensure that the derivatives of the velocity equal to zero on the Earth surface and a sufficiently rapid decay for all thermodynamic quantities as the vertical coordinate z approaches to infinity. In other words, the impenetrability conditions make sure the boundedness of the mass, energy, and momentum in the air column. They also provide the necessary conditions for the convergence of integrals.

The source terms F_1 and F_2 include basically the (turbulent) viscosity, heat conductivity, and may be something else. In fact, the form of F_1 and F_2 for real geophysical model depends on the model considered, in any case these terms are believed to be small and to have a regularizing effect. Indeed, the solution to system (2.1)–(2.3) (i.e. the compressible Euler system) generally loses smoothness within a finite time with formation of shocks, and it needs to introduce some smoothing terms to avoid this phenomenon not inherent to the atmospheric motion.

For three-dimensional equations it is the practice to introduce the entropy function s , which is connected with the pressure and density through the state equation

$$p = a e^s \rho^{\tilde{\gamma}},$$

with a constant $a > 0$.

We can also introduce the 2D entropy $S(t, x)$ connecting with ϱ and P similarly to (2), where we use γ instead of $\tilde{\gamma}$. Then instead of equation (2.3) we get the equation for the entropy

$$\partial_t S + (\mathbf{U}, \nabla S) = F_3, \quad (2.4)$$

with a source F_3 , see [30] for details.

For the sake of simplicity and to make our idea clear in this paper we will consider only the barotropic case of constant entropy S_0 , in other words, the state equation is

$$P = C \rho^\gamma, \gamma > 1, \quad C = \text{const} > 0, \quad (2.5)$$

and (2.4) is assumed to hold identically with $F_3 = 0$.

Thus, the system under consideration is reduced to two equations (2.1), (2.2).

It will be convenient to introduce a new variable $\pi = P^{\frac{\gamma-1}{\gamma}}$. For the new unknown variables $\pi(t, x)$, $\mathbf{U}(t, x)$ we obtain the system

$$\begin{aligned} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{U} + l L \mathbf{U} + c_0 \nabla \pi &= F, \\ \partial_t \pi + (\nabla \pi \cdot \mathbf{U}) + (\gamma - 1) \pi \operatorname{div} \mathbf{U} &= 0, \end{aligned} \quad (2.6)$$

with a new source term $F(t, x)$ and $c_0 = \frac{\gamma}{\gamma-1} C^{\frac{1}{\gamma}}$.

3. "Frozen" vortex: a general approach. Let us change the coordinate system in such a way that the origin of the new system $\mathbf{x} = (x_1, x_2)$ is located at a point $\mathbf{X}(t) = (X_1(t), X_2(t))$ (here and below we use the bold font for \mathbf{x} to denote the local coordinate system). Now $\mathbf{U} = \mathbf{u} + \mathbf{V}$, where $\mathbf{V}(t) = (V_1(t), V_2(t)) = (\dot{X}_1(t), \dot{X}_2(t))$. Thus, we obtain a new system

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \dot{\mathbf{V}} + l(\mathbf{X}(t) + \mathbf{x}) L(\mathbf{u} + \mathbf{V}) + c_0 \nabla \pi = F, \quad (3.1)$$

$$\partial_t \pi + \nabla \pi \cdot \mathbf{u} + (\gamma - 1) \pi \operatorname{div} \mathbf{u} = 0. \quad (3.2)$$

Given a vector \mathbf{V} , the trajectory can be found by integrating the system

$$\dot{X}_1(t) = V_1(t), \quad \dot{X}_2(t) = V_2(t). \quad (3.3)$$

Below we look for a solution with special properties, namely, a steady divergency free vortex "frozen" into a certain exterior pressure field. Together with this we study *a class of sources F allowing the existence of such solutions*. Let us stress that we do not prescribe specific boundary conditions and only *assume* that

$$\mathbf{u} \text{ does not depend of } t, \quad \mathbf{u}(0) = 0 \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0.$$

This means that there exists a potential $\Phi(x_1, x_2)$ such that

$$\mathbf{u} = \nabla_{\perp} \Phi = (\Phi_{x_2}, -\Phi_{x_1}). \quad (3.4)$$

Thus, the velocity field satisfies the Cauchy-Riemann conditions

$$\partial_{x_1} u_1 = \partial_{x_2} u_2, \quad \partial_{x_2} u_1 = -\partial_{x_1} u_2.$$

In [11] it is proved that for the case of l -plane this condition is necessary for the existence of a stable vortex.

Further, we *assume* that

$$\pi = \pi_0(x_1, x_2) + \pi_1(t, x_1, x_2), \quad (3.5)$$

where

$$(\nabla \pi_0, \nabla_\perp \Phi) = 0. \quad (3.6)$$

We call the time-dependent component π_1 a *bearing field of pressure*. It will be important for us that the gradient of the bearing field is rather small. The time-independent part of pressure π_0 relates to the vortex itself.

Let A be an appropriate (2×2) matrix which depends on the situation to be discussed below. We consider the following equation for the potential of velocity Φ and the steady part of pressure π_0 :

$$(\nabla_\perp \Phi, \nabla) \nabla_\perp \Phi + A \nabla_\perp \Phi + c_0 \nabla \pi_0 = 0. \quad (3.7)$$

Let us suppose that we succeed to solve system (3.7), (3.6). Then from (3.2) we get a linear equation for π_1 :

$$\partial_t \pi_1 + \nabla \pi_1 \cdot \mathbf{u} = 0, \quad (3.8)$$

which can be solved for any initial condition $\pi_1(0, \mathbf{x})$.

Further, from (3.1) we obtain

$$\dot{\mathbf{V}}(t) + l(\mathbf{X}(t) + \mathbf{x}) L \mathbf{V}(t) + (l(\mathbf{X}(t) + \mathbf{x}) L - A) \mathbf{u}(\mathbf{x}) + c_0 \nabla \pi_1(t, \mathbf{x}) = F, \quad (3.9)$$

or

$$\ddot{\mathbf{X}}(t) + l(\mathbf{X}(t) + \mathbf{x}) L \dot{\mathbf{X}}(t) + (l(\mathbf{X}(t) + \mathbf{x}) L - A) \mathbf{u}(\mathbf{x}) + c_0 \nabla \pi_1(t, \mathbf{x}) = F. \quad (3.10)$$

Let us denote

$$Q = -(l(\mathbf{X}(t)) - l(\mathbf{X}(t) + \mathbf{x})) L \dot{\mathbf{X}}(t) + (l(\mathbf{X}(t) + \mathbf{x}) L - A) \mathbf{u}(\mathbf{x}) - c_0 \left[\nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} - \nabla \pi_1(t, \mathbf{x}) \right]. \quad (3.11)$$

From (3.10) and (3.11) we get

$$\ddot{\mathbf{X}}(t) + l(\mathbf{X}(t)) L \dot{\mathbf{X}}(t) + c_0 \nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} = F - Q. \quad (3.12)$$

If F equals Q , then the position of the center of vortex under consideration can be found from the following equation:

$$\ddot{\mathbf{X}}(t) + l(\mathbf{X}(t)) L \dot{\mathbf{X}}(t) + c_0 \nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} = 0. \quad (3.13)$$

Thus, (3.1) can be rewritten as

$$\begin{aligned} & (\mathbf{u} \cdot \nabla) \mathbf{u} + A\mathbf{u} + c_0 \nabla \pi_0 + \\ & \ddot{\mathbf{X}}(t) + l(\mathbf{X}(t))L\dot{\mathbf{X}}(t) + c\nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} = \\ & F - Q, \end{aligned} \quad (3.14)$$

the terms in the first row depend only on the space variable \mathbf{x} , the terms in the second row depend only on t , therefore for the case $F - Q = 0$ we obtain a complete separation of variables for solution satisfying (2.6), (3.4), (3.5), (3.6), (3.8), (3.13).

If $F - Q$ is not zero, however it is small (for example, the uniform norm $\|F - Q\| < \delta$), we can talk about a " δ - approximate" separation of variables, $F - Q$ plays a role of the discrepancy.

DEFINITION 3.1. *We call a couple $(\pi(t, x), \mathbf{U}(t, x))$ the δ - approximate solution to system (2.6) in a domain $D(\delta) \in \mathbb{R}_+ \times \mathcal{M}$, if (π, \mathbf{U}) satisfies (2.6) with a discrepancy, whose uniform norm is smaller than δ for any $(t, x) \in D(\delta)$.*

In fact we have to choose the matrix A in such a way as to ensure the smallness of the discrepancy term $F - Q$. Since we are interested in studying the behavior of solution near the origin of the moving coordinates system, we expand in (3.11) the known functions $l(\mathbf{x})$, $\mathbf{u}(\mathbf{x})$ and $\pi_1(t, \mathbf{x})$ in the origin and get $Q = O(|\mathbf{x}|)$ provided $\dot{\mathbf{X}}(t)$ keeps boundedness. Basically a solution to the nonlinear equation (3.12) blows up in a finite time t_* , this entails an unboundness of $\dot{\mathbf{X}}(t)$. Thus, our considerations are valid for $t < t_*$, until the formation of the shock wave in a neighborhood of the trajectory $\mathbf{X}(t)$ of the moving coordinate system.

Let us list once more the steps of our method. First we should solve (3.6), (3.7), then set an initial distribution for the bearing pressure field $\pi_1(0, x_1, x_2)$, then find the respective time-dependent pressure component $\pi_1(t, x_1, x_2)$ and at last integrate (3.10) to find exact (for $F = Q$) or approximate (for small $F - Q$) trajectory of the exact (or δ - approximate) vortex solution.

Thus, the main problem is to solve the system (3.6), (3.7), with respect to unknown scalar functions Φ and π_0 (may be under suitable boundary conditions).

Let us find a necessary condition for the function Φ that allows it to be a part of solution of (3.6), (3.7). We take the inner product of (3.7) and $\nabla_\perp \Phi$ and get the master equation

$$\nabla_{x_1 x_2} \Phi ((\nabla_{x_1} \Phi)^2 - (\nabla_{x_2} \Phi)^2) + (A \nabla_\perp \Phi, \nabla_\perp \Phi) = \nabla_{x_1} \Phi \nabla_{x_2} \Phi (\nabla_{x_2 x_2} \Phi - \nabla_{x_1 x_1} \Phi). \quad (3.15)$$

If the solution of (3.15) satisfies the identity

$$\nabla \times ((\nabla_\perp \Phi, \nabla) \nabla_\perp \Phi + A \nabla_\perp \Phi) = 0, \quad (3.16)$$

then one can find π_0 such that the couple (Φ, π_0) solves (3.6), (3.7).

Let us summarize the results of this section.

THEOREM 3.2. *Let $\Phi(\mathbf{x})$ be a solution to equation (3.15), satisfying condition (3.6) with a matrix A with smooth coefficients dependent only on space variables. Further, let $\pi_0(\mathbf{x})$ be a solution to (3.7), $\pi_1(t, \mathbf{x})$ be a solution to (3.8) with $\mathbf{u} = \nabla_\perp \Phi$, $\mathbf{u}(0) = 0$, and $\mathbf{X}(t)$ be a solution to (3.13). Then the couple (π, \mathbf{U}) , where*

$$\pi(t, x) = \pi_0(x - \mathbf{X}(t)) + \pi_1(t, x - \mathbf{X}(t)), \quad \mathbf{U}(t, x) = \dot{\mathbf{X}}(t) + \mathbf{u}(x - \mathbf{X}(t)) \quad (3.17)$$

solve system (2.6) with $F = Q$, Q is given by (3.11).

THEOREM 3.3. *We assume $F = 0$. Let $\Phi, \pi_0, \pi_1, \mathbf{X}(t)$ be as in Theorem 3.2, moreover, they are classical solutions to the respective equations. Assume $|\dot{\mathbf{X}}(t)|$ to be*

bounded for $t \leq T$. Then there exists a neighborhood $D(\delta)$ of the trajectory $x = \mathbf{X}(t)$ such that (3.17) is the δ -approximate solution to system (2.6) for $t \leq T$. The dimensions of $D(\delta)$ depend on derivatives of known functions, namely, $|\nabla l(x)|$, $|\nabla \pi(t, x)|$, $|\mathbf{h}|$, where $\mathbf{h} = (h_1, h_2)$, $h_j = \sum_{i=1}^2 (lL - A)_{ij} u_i$, $j = 1, 2$, $\mathbf{u} = (u_1, u_2)$.

Proof. The proof of Theorem 3.3 follows from estimating the term Q given by (3.11). \square

Below we dwell on important for geophysics cases of application of the method.

4. Steady vortices on a plane. The "plane" models are used in meteorology to describe the processes of small and medium scale, where the curvature of the Earth surface does not play a crucial role and the change of the Coriolis parameter can be modeled in a simplest way.

Let $\mathbf{x}_0 = (x_{01}, x_{02})$ be a point on the Earth surface, φ_0 be the latitude of some fixed point \mathbf{x}_0 . The simplest possible "plane" model is the so called l -plane model, where the Coriolis parameter l is treated as a constant: $l = l_0 = 2\Omega \sin \varphi_0$, Ω is the vertical component of the angular velocity of the Earth rotation. The model, that is believed to be more adequate to describe the weather processes, is the β -plane model, where the Coriolis parameter l is approximated by a linear function: $l = l_0 + \beta x_2$, the constant $\beta = \frac{2\omega_3}{R} \cos \varphi_0$, where R is the radius of the Earth. In fact, in the l -plane model we neglect the parameter β , which is much more smaller than l_0 . For example, for $\varphi_0 = 30^\circ$, $l_0 \approx 7.3 \times 10^{-5} \text{s}^{-1}$, $\beta \approx 2 \times 10^{-11} \text{s}^{-1}$, where we used the values $R = 6.4 \times 10^6 \text{m}$ and $\omega_3 = 7.3 \times 10^{-5} \text{rad/s}$.

It can be readily checked that for $l = l_0 = \text{const}$ every potential

$$\Phi = \Phi(r), \quad r = \sqrt{x_1^2 + x_2^2},$$

solves the master equation (3.15) with $A = l_0 L$. We can get different shapes of vortices choosing different solutions.

For the β -plane model we will use the master equation with the same matrix $A = l_0 L$. Recall that the choice of this matrix can be different, nevertheless it is caused by a desire to make the discrepancy term $F - Q$ as smaller as possible.

4.1. Example 1: steady vortex on the l -plane (linear profile of velocity). We begin from the simplest case where for $F = Q = 0$ we have the complete separation of variables in equation (3.14). Here we can obtain an exact solution. Indeed, if we choose $\Phi = \frac{b_0}{2} (x_1^2 + x_2^2)$, $b_0 = \text{const}$, we get the velocity field with a linear profile

$$u_1 = b_0 x_2, \quad u_2 = -b_0 x_1. \quad (4.1)$$

Further, if we choose the initial data for π_1 as a linear function of the space variables, $\pi_1(0, x) = M_{10} x_1 + M_{20} x_2 + K_0$, $M_{10}, M_{20}, K_0 = \text{const}$, we get

$$\pi_1(t, x) = M_1(t) x_1 + M_2(t) x_2 + K(t), \quad (4.2)$$

where

$$M_1(t) = M_{10} \cos b_0 t + M_{20} \sin b_0 t, \quad M_2(t) = M_{20} \cos b_0 t - M_{10} \sin b_0 t. \quad (4.3)$$

The trajectory obeys the equation

$$\ddot{\mathbf{X}}(t) + l_0 L \dot{\mathbf{X}}(t) + c_0 \mathbf{M}(t) = 0, \quad (4.4)$$

where $\mathbf{M}(t) = (M_1(t), M_2(t))$.

The solution can be found explicitly:

$$\begin{aligned}
X_1(t) = & X_1(0) + \frac{V_2(0)}{l} + \frac{c_0 M_{10}}{b_0 l} \\
& + \left(\frac{V_1(0)}{l} - \frac{c_0 M_{20}}{l(b_0 - l)} \right) \sin lt - \left(\frac{V_2(0)}{l} + \frac{c_0 M_{10}}{l(b_0 - l)} \right) \cos lt \\
& + \frac{c_0 M_{20}}{b_0(b_0 - l)} \sin b_0 t + \frac{c_0 M_{10}}{b_0(b_0 - l)} \cos b_0 t, \\
X_2(t) = & X_2(0) - \frac{V_1(0)}{l} + \frac{c_0 M_{20}}{b_0 l} \\
& + \left(\frac{V_2(0)}{l} + \frac{c_0 M_{10}}{l(b_0 - l)} \right) \sin lt + \left(\frac{V_1(0)}{l} - \frac{c_0 M_{20}}{l(b_0 - l)} \right) \cos lt \\
& - \frac{c_0 M_{10}}{b_0(b_0 - l)} \sin b_0 t + \frac{c_0 M_{20}}{b_0(b_0 - l)} \cos b_0 t,
\end{aligned}$$

(for $l \neq b_0$). This is a superposition of two circular motions: the respective frequencies are $\frac{2\pi}{l_0}$ and $\frac{2\pi}{b_0}$.

On this way we obtain the same solution as in [29], [30]. The above formulae were used in [30] to imitate and forecast the trajectories of tropical cyclones based on real observational data. In fact, we are going to show that a further sophistication of model do not implies significant difference in the behavior of the vortex trajectory, at least for the "realistic" (relative to meteorological data) values of parameters.

4.2. Example 2: approximately steady vortex on the β - plane (linear profile of velocity). Let us choose again $A = l_0 L$, $\Phi = \frac{b_0}{2} (x_1^2 + x_2^2)$, $\pi_1(0, x) = M_{10} x_1 + M_{20} x_2 + K_0$, $M_{10}, M_{20}, K_0 = \text{const.}$ Thus, (4.1) gives the velocity field and (4.2) gives the background pressure field π_1 as in Ex.1.

To obtain the solution of the form of the steady vortex we need to set $F = Q$, where Q (see (3.11)) has the form

$$Q = \beta x_2 L (\dot{\mathbf{X}}(t) + \mathbf{u}(\mathbf{x})) + \beta X_2(t) L \mathbf{u}(\mathbf{x}). \quad (4.5)$$

To find the trajectory of the steady vortex we have to integrate the equation (3.12)

$$\ddot{\mathbf{X}}(t) + (l_0 + \beta X_2(t)) L \dot{\mathbf{X}}(t) + c_0 \mathbf{M}(t) = 0. \quad (4.6)$$

If $F = 0$, we can consider Q as a vector of discrepancy, and due to the smallness of β for real meteorological models the value of Q is essentially small until the blow up time of the respective solution to the ODE (4.6).

4.3. Example 3: steady localized vortex on the l - plane. As was mentioned before the velocity field (4.1) rises at infinity and is not realistic. Thus it is natural to consider the vortex in some sense localized in the space. We will call the vortex "localized", if the velocity profile looks like the observational data (e.g. [34], see Fig.4.1). This kind of initial data is used for the majority of numerical computations simulating tropical cyclones. It is catching to select the form of the vortex basing on the experimental data, for example, in the class of potential functions

$$\Phi(r) = \frac{a}{(1 + \sigma r^2)^k}, \quad a > 0, \sigma > 0, k > 0. \quad (4.7)$$

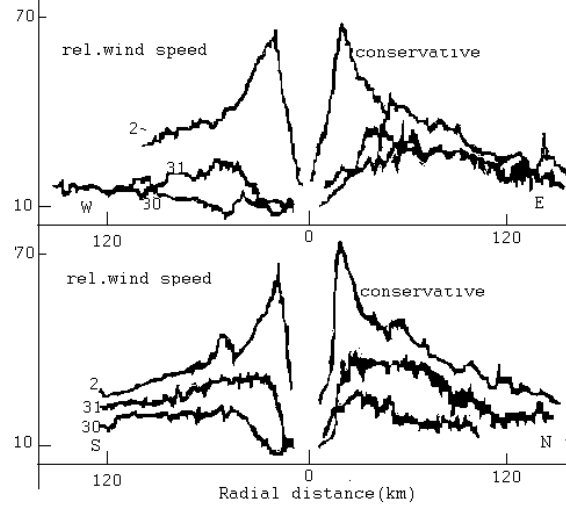


FIGURE 4.1. The tangential component of (relative) velocity in $\text{m} \cdot \text{s}^{-1}$ on different stage of the typhoon formation (August 30, August 31, September 2). The hurricane Anita, North Atlantic, 1977 [34]. The horizontal scale is in kilometers.

Indeed, the dashed line in Fig.4.2 presents the graph of the modulus of the radial component of the corresponding velocity field for $k = \frac{1}{2}$, $a = 7.8 \times 10^6 \text{ m}^2/\text{s}$, $\sigma = 5 \times 10^{-10} \text{ m}^{-2}$. The parameters give a correct maximum of velocity and a realistic decay of the velocity at infinity. Nevertheless, one can remember that in our model we use the averaged over the height velocity and to construct the graph analogous to Fig.4.1 for averaged velocity we need to analyze the experimental data for different height levels (that we do not have at our disposal). Thus, the profile of velocity for averaged data may be different from those that presented in Fig.4.1. At least, the value of amplitude a should be much more smaller due to the different direction of rotation of the air for lower and upper parts of cyclone [34]. Moreover, if we use the pattern (4.7), we cannot solve explicitly equation (3.8) for the background pressure field. Therefore we use a more localized (exponentially decaying) solution to the master equation (3.15) with $A = l_0 L$:

$$\Phi = -B_0 e^{-\frac{\sigma}{2}(x_1^2 + x_2^2)}, \quad (4.8)$$

here B_0 is a constant, σ is a positive constant. The respective velocity field

$$u_1 = B_0 \sigma x_2 e^{-\frac{\sigma}{2}(x_1^2 + x_2^2)}, \quad u_2 = -B_0 \sigma x_1 e^{-\frac{\sigma}{2}(x_1^2 + x_2^2)} \quad (4.9)$$

vanishes at infinity and has the same asymptotics as (4.1) at the origin, with $b_0 = B_0 \sigma$. The solid line in Fig.4.2 shows the modulus of the radial component of velocity for $B_0 = 3.5 \times 10^{-6} \text{ m}^2/\text{s}$, $\sigma = 10^{-9} \text{ m}^{-2}$.

We are going to show that for the values of parameters relevant in the meteorology such strict localization does not lead to a big difference in trajectories of the localized and non-localized vortices.

It can be checked that condition (3.16) is satisfied and therefore we can find the

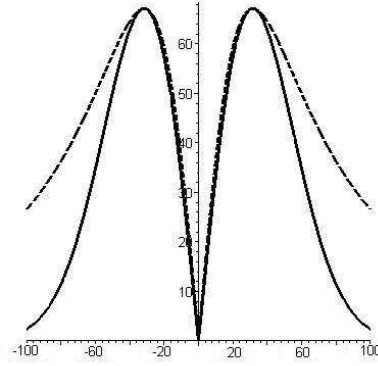


FIGURE 4.2. *The tangential component of (relative) velocity in $m \cdot s^{-1}$ for a localized vortex: exponential decay (solid line) and power decay (dashed line). The horizontal scale is in kilometers.*

scalar function π_0 . Namely,

$$\pi_0(x_1, x_2) = \frac{1}{c_0} \left(\frac{1}{2} B_0^2 \sigma e^{-\sigma(x_1^2 + x_2^2)} - l_0 B_0 e^{-\frac{\sigma}{2}(x_1^2 + x_2^2)} \right). \quad (4.10)$$

Further, equation (3.8) can be explicitly solved, its general solution is

$$\pi_1 = G \left(x_1^2 + x_2^2, t - \frac{1}{B_0 \sigma} e^{\frac{\sigma}{2}(x_1^2 + x_2^2)} \arctan \left(\frac{x_1}{x_2} \right) \right),$$

with any differentiable function G . For example, for initial background field

$$\pi_1(0, x_1, x_2) = R_0 + \phi(x_1, x_2) (M_{10} x_1 + M_{20} x_2 + K_0), \quad (4.11)$$

$$R_0, M_{10}, M_{20}, K_0 = \text{const}, \quad \phi_0(x_1, x_2) = e^{-\sigma_0(x_1^2 + x_2^2)},$$

we get

$$\begin{aligned} \pi_1(t, x_1, x_2) = & R_0 + \phi_0(x_1, x_2) (K_0 + M_{10} (x_1 \cos(\phi(x_1, x_2) B_0 \sigma t) - x_2 \sin(\phi(x_1, x_2) B_0 \sigma t)) + \\ & M_{20} (x_2 \cos(\phi(x_1, x_2) B_0 \sigma t) + x_1 \sin(\phi(x_1, x_2) B_0 \sigma t))), \end{aligned}$$

with $\phi(x_1, x_2) = e^{-\frac{\sigma}{2}(x_1^2 + x_2^2)}$.

The trajectory of the vortex can be found by integration from equation (3.12):

$$\ddot{\mathbf{X}}(t) + l_0 L \dot{\mathbf{X}}(t) + c_0 \nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} = 0. \quad (4.12)$$

It can be checked that

$$\nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} = (M_1(t), M_2(t)),$$

with the same $M_1(t)$ and $M_2(t)$ as in (4.3).

Thus, equation (4.4) that governs the trajectory of the non-localized vortex for this specific background pressure field coincides with (4.12).

The discrepancy term (for $F = 0$) here is

$$Q = c_0 \left[\nabla \pi_1(t, \mathbf{x}) - \nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} \right], \quad (4.13)$$

it depends only on the properties of the background pressure field π_1 . From the explicit expression for π_1 , presented above, one can see that Q is bounded and its norm vanishes at every point of plane as the parameters of initial slope of the background pressure field M_{10} and M_{20} tend to zero.

4.4. Example 4: steady localized vortex on the β - plane. We use in this case the same master equation as in Ex.3 and its solution (4.8). Moreover, we use the same initial background pressure field $\pi_1(0, \mathbf{x})$ as in Ex.3. The equation (3.12) for the trajectory of the steady vortex is

$$\ddot{\mathbf{X}}(t) + (l + \beta X_2(t)) L \dot{\mathbf{X}}(t) + c_0 \nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} = 0, \quad (4.14)$$

this is the same equation as (4.6), the discrepancy Q (for $F = 0$) is

$$Q = \beta x_2 L (\dot{\mathbf{X}}(t) + \mathbf{u}(\mathbf{x})) + \beta X_2(t) L \mathbf{u}(\mathbf{x}) + c_0 \left[\nabla \pi_1(t, \mathbf{x}) - \nabla \pi_1(t, \mathbf{x}) \Big|_{x=0} \right].$$

It can be considered as a small one due to the reasons described in Examples 2 and 3.

4.5. Comparison of trajectories for the l - plane and the β - plane models. We compared the trajectories of vortices corresponding to approximate solutions under values of parameters close to the realistic ones. Namely, we set $\Omega = 7.3 \times 10^{-5} \text{s}^{-1}$, the geographical latitude $\phi_0 = 30^\circ$, the Earth radius $R = 6.39 \times 10^6 \text{m}$, $c_0 = 0.1$ (appropriate dimension), $B_0 = 3.5 \times 10^3 \text{m}^2/\text{s}$, $\sigma = 10^{-9} \text{m}^{-2}$, $\sigma_0 = 8 \times 10^{-13} \text{m}^{-2}$, $M_0 = 2 \times 10^{-3} \text{s}^{-1}$, $N_0 = 10^{-3} \text{s}^{-1}$, $X_1(0) = X_2(0) = 0$, $V_1(0) = -1 \text{m/s}$, $V_2(0) = 1 \text{m/s}$.

The result is the following: the difference between the l - plane and the β - plane models is very small for these parameters up to 2 days period of computation, further the difference increases, however the behavior of the both trajectories remains similar (see Fig.4.3).

4.6. Computational experiments with localized vortices on the plane: influence of the discrepancy term. As it was shown before, if the discrepancy term is zero, the vortex is steady and it moves according to (3.12). Nevertheless, let us study the influence of the discrepancy on the shape of vortex. It is natural to expect that a big discrepancy destroys the vortex, whereas a negligible one only modifies it.

Figs.4.4 – 4.6 present the result of computations for the l - plane and the β - plane models. Here $\varphi_0 = 30^\circ$, $l_0 \approx 7.3 \times 10^{-5} \text{s}^{-1}$, $\beta \approx 2 \times 10^{-11} \text{s}^{-1}$, $u_{10} = V_1(0) = u_{20} = V_2(0) = 10 \text{m/s}$. The initial position of vortex is the origin. The parameter B_0 , σ , c_0 are as in the Sec.4.5. The localized vortex is given initially as (4.10),

$$u_1 = u_{10} + B_0 \sigma x_2 e^{-\frac{\sigma}{2}(x_1^2 + x_2^2)}, \quad u_2 = u_{20} - B_0 \sigma x_1 e^{-\frac{\sigma}{2}(x_1^2 + x_2^2)}. \quad (4.15)$$

Fig.4.4 corresponds to the pressure for $\pi_1(0, x_1, x_2) = 0$ (zero bearing field and discrepancy), the first picture presents the level lines for the initial data (common for all figures), the second one corresponds to the vortex computed for approximately 4 days. Fig.4.5 corresponds to the vortex on the l - plane in the bearing field. The

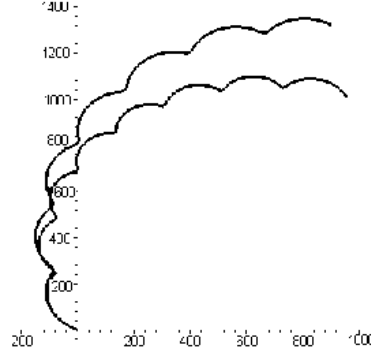


FIGURE 4.3. Trajectories of vortices on l -plane (upper curve) and β -plane for 7 days period. The space scale is in kilometers.

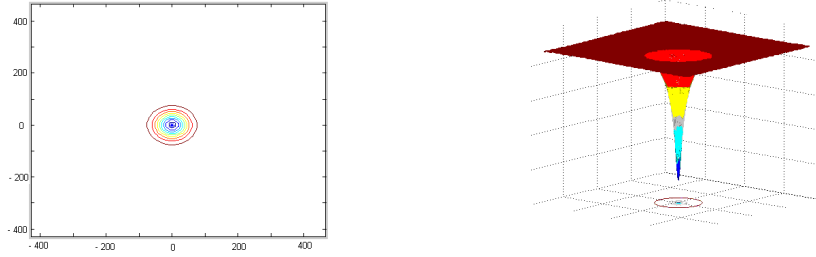
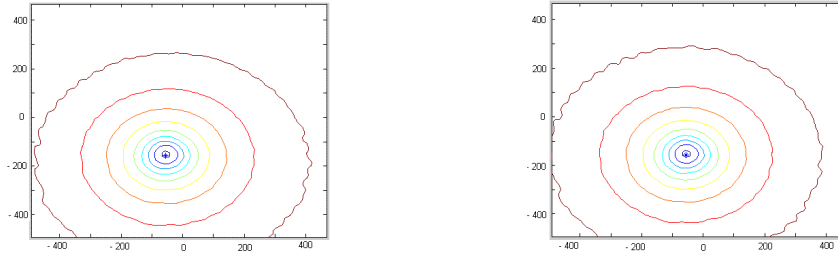


FIGURE 4.4. The l -plane case without bearing field, initial vortex, $T = 0$ and $T \approx 96h$.

parameters of "inclination" of the bearing field are $M_{10} = -10^{-5}$, $M_{20} = 10^{-5}$ (weak bearing field) for the first picture and $M_{10} = -10^{-5}$, $M_{20} = 10^{-5}$ (stronger bearing field) for the second picture. Computations are made for one day, the space scale is the same for both pictures in Fig.4.5 and for the second picture in Fig. 4.4. We can see that the vortex preserves its initial shape for the null bearing field, it modifies its form and enlarges in a weak bearing field and tends to disappear in a strong bearing field. Fig. 4.6 corresponds to the initial bearing field of form (4.11) with $\sigma_0 = 10^{-13}$ (here and below the dimension is appropriate) for the l -plane and the β -plane models. The parameters of "inclination" of the bearing field are $M_{10} = -10^{-5}$, $M_{20} = 10^{-5}$ (weak bearing field, approximately 2 days period). The blue asterisk corresponds to the theoretical position of vortex computed according to (4.12) and (4.14) for l -plane and β -plane models, respectively. The position of the center of vortex, obtained numerically correlates precisely with the theoretical results. The computations were made by a modified Lax-Wendroff scheme, the method is second order accurate in both space and time variables [27], [39], [20]. We apply the scheme to the system in the conservation form written compactly as follows:

$$\frac{\partial U}{\partial t} = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + S,$$


 FIGURE 4.5. *The pressure, weak and stronger bearing fields, $T \approx 1$ days*

 FIGURE 4.6. *The l -plane and the β - plane cases, respectively, for $T \approx 48$ h, weak bearing field.*

where

$$U = \begin{pmatrix} \varrho \\ \varrho u_1 \\ \varrho u_2 \end{pmatrix}$$

$$F = \begin{pmatrix} -\varrho u_1 \\ -\varrho u_1^2 - \frac{9c_0}{16} \varrho^{\frac{16}{7}} \\ -\varrho u_1 u_2 \end{pmatrix}$$

$$G = \begin{pmatrix} -\varrho u_2 \\ -\varrho u_1 u_2 \\ -\varrho u_2^2 - \frac{9c_0}{16} \varrho^{\frac{16}{7}} \end{pmatrix}$$

$$S = \begin{pmatrix} 0 \\ l u_2 \\ -l u_1 \end{pmatrix}$$

Suppose that the solution domain in $2D$ is divided into rectangular cells. Let $U_{i,j}^n = U((i - \frac{1}{2})\Delta x, (j - \frac{1}{2})\Delta y, n\Delta t)$ be the value of U in the center of (x_i, y_j) at the time

level t^n . Let us denote

$$\begin{aligned}\bar{f}_{i,j} &= \frac{1}{4}(f_{i-\frac{1}{2},j-\frac{1}{2}} + f_{i-\frac{1}{2},j+\frac{1}{2}} + f_{i+\frac{1}{2},j-\frac{1}{2}} + f_{i+\frac{1}{2},j+\frac{1}{2}}), \\ (\delta_x f)_{i,j} &= \frac{1}{\Delta x} \overline{(f_{i+\frac{1}{2},j} - f_{i-\frac{1}{2},j})^y}, \\ (\delta_y f)_{i,j} &= \frac{1}{\Delta y} \overline{(f_{i,j+\frac{1}{2}} - f_{i,j-\frac{1}{2}})^x}, \\ \bar{f}_{i,j}^x &= \frac{1}{2}(f_{i+\frac{1}{2},j} + f_{i-\frac{1}{2},j}), \\ \bar{f}_{i,j}^y &= \frac{1}{2}(f_{i,j+\frac{1}{2}} + f_{i,j-\frac{1}{2}}).\end{aligned}$$

The explicit two-step scheme is given by

$$\begin{aligned}U_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}} &= \bar{U}_{i+\frac{1}{2},j+\frac{1}{2}}^n + \frac{\Delta t}{2}(\delta_x F + \delta_y G + \bar{S})_{i+\frac{1}{2},j+\frac{1}{2}}^n, \\ U_{i,j}^{n+1} &= U_{i,j}^n + \Delta t(\delta_x F + \delta_y G + \bar{S})_{i,j}^{n+\frac{1}{2}}.\end{aligned}$$

The computations were performed on a (240×240) uniform grid with the space step $\Delta x = \Delta y = 0.64$ which corresponds to 12.8 km and the time step $\Delta t = 0.0005$ which corresponds to 10 sec of the real time. We use the Neumann boundary condition set sufficiently far from the vortex domain. Nevertheless, it is possible to use more sophisticated non-reflecting boundary conditions [16] and introduce an artificial viscosity to damp the oscillations [26].

5. Steady vortices on the sphere. In Sections 2 and 3 we mentioned that all equations, e.g. (2.1) – (2.3), (2.6), (3.1), (3.2), can be written in curvilinear coordinates, in this case we mean by derivatives the covariant derivatives with respect to the metric of a Riemannian manifold.

For example let us rewrite system (2.1) – (2.2) avoiding the tensor notation in the spherical coordinates, making clear the influence of curvature of the space, as this system usually appears in the geophysical textbooks (e.g.[25]):

$$\varrho \left(\frac{dU}{dt} - \left(2\Omega + \frac{U}{R \cos \tilde{\phi}} \right) V \sin \tilde{\phi} + \frac{1}{R \cos \tilde{\phi}} \frac{\partial P}{\partial \tilde{\lambda}} \right) = f_1,$$

$$\varrho \left(\frac{dV}{dt} + \left(2\Omega + \frac{U}{R \cos \tilde{\phi}} \right) U \sin \tilde{\phi} + \frac{1}{R} \frac{\partial P}{\partial \tilde{\phi}} \right) = f_2,$$

$$\frac{\partial}{\partial t}(\varrho \cos \tilde{\phi}) + \frac{\partial}{\partial \tilde{\lambda}} \left(\frac{\varrho U \cos \tilde{\phi}}{R} \right) + \frac{\partial}{\partial \tilde{\phi}} \left(\frac{\varrho V \cos \tilde{\phi}}{R} \right) = 0,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{U}{R \cos \tilde{\phi}} \frac{\partial}{\partial \tilde{\lambda}} + \frac{V}{R} \frac{\partial}{\partial \tilde{\phi}}$, R is the radius of the Earth.

Here we use the established notation $(\tilde{\lambda}, \tilde{\phi})$ for the geographical longitude and latitude, $\tilde{\lambda} \in (-\pi, \pi)$, $\tilde{\phi} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (we mark coordinates by the character tilde to avoid confusion with the moving coordinate system, where the latitude and longitude are denoted as (λ, ϕ)). Further, (U, V) and (f_1, f_2) stand for the components of

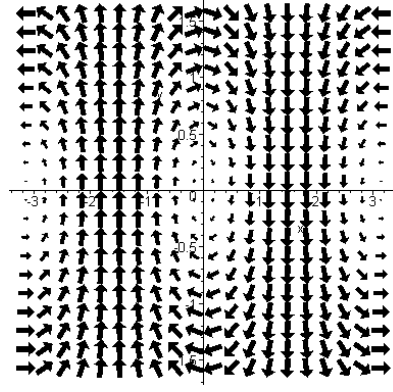


FIGURE 5.1. The periodic field of velocity corresponding to $\Phi = -\cos \lambda \cos \phi$. The space scale is in radians.

velocity and the vector of exterior forces F , respectively. The Coriolis parameter is $l = 2\Omega \sin \tilde{\phi}$, Ω is again the vertical component of the angular velocity of the Earth.

The master equation (3.15) now has the form (in the moving coordinate system)

$$\begin{aligned} & \frac{\partial^2 \Phi}{\partial \lambda \partial \phi} \left(\left(\frac{\partial \Phi}{\partial \phi} \right)^2 - \frac{1}{\cos^2 \phi} \left(\frac{\partial \Phi}{\partial \lambda} \right)^2 \right) + \\ & \left(\frac{1}{\cos^2 \phi} \frac{\partial^2 \Phi}{\partial \lambda^2} - \frac{\partial^2 \Phi}{\partial \phi^2} \right) \frac{\partial \Phi}{\partial \lambda} \frac{\partial \Phi}{\partial \phi} - \frac{\sin \phi}{\cos^3 \phi} \left(\frac{\partial \Phi}{\partial \lambda} \right)^3 = 0, \end{aligned} \quad (5.1)$$

Equation (5.1) has a solution of the form

$$\Phi(\lambda, \phi) = \Phi(\cos \phi (a \sin \lambda + b \cos \lambda) + h \sin \phi), \text{ where } a, b, h \text{ are constants.} \quad (5.2)$$

In particular, one can take

$$\Phi(\lambda, \phi) = b \cos \lambda \cos \phi, \quad b = \text{const}, \quad (5.3)$$

to obtain the zero of the respective velocity field at the origin of the coordinate system. Thus, this field is

$$u(\lambda, \phi) = -b \cos \lambda \sin \phi, \quad v(\lambda, \phi) = b \sin \lambda. \quad (5.4)$$

This field plays the similar fundamental role in the case of the spherical geometry as the velocity with linear profile in the case of the plane. It is easy to see that $(u, v) \sim (-b\phi, b\lambda)$ at the point $\lambda = \phi = 0$. To obtain a cyclonic (clockwise) vorticity at the origin, we can set $b < 0$. Fig.5.1 presents the field of velocity for $b = -1$. The velocity of form (5.4) was used for testing the numerical methods, e.g. [18],[23].

If $a^2 + b^2 \neq 0$, condition (3.16) can be satisfied only for $A = l_0 L$, $l_0 = \text{const}$. For example, the pressure field π_0 , corresponding to 5.4, is

$$\pi_0(\lambda, \phi) = C - \frac{b \cos \lambda \cos \phi}{2c_0} (b \cos \lambda \cos \phi + 2l_0), \quad C = \text{const}. \quad (5.5)$$

The expansion at the origin has a form

$$\pi_0(\lambda, \phi) = C - \frac{b(b + l_0)}{2c_0} - \frac{b(b + l_0)}{2c_0} (\lambda^2 + \phi^2) + o((\lambda^2 + \phi^2)). \quad (5.6)$$

Equation (3.8) can be also solved here; a rather cumbersome solution is expressed in elliptic functions $E_F(z, k)$ and $E_P(z, \nu, k)$ (see [1]):

$$\pi_1(t, \lambda, \phi) = G \left(\cos \phi \cos \lambda, bt + \cos \phi \sin 2\lambda (E_F(z, k) - 2E_P(z, k, k)) \right), \quad (5.7)$$

with arbitrary differentiable function G . Here

$$E_F(z, k) = \int_0^z \frac{1}{\sqrt{1-\xi^2} \sqrt{1-k^2\xi^2}} d\xi,$$

$$E_P(z, \nu, k) = \int_0^z \frac{1}{(1-\nu\xi^2) \sqrt{1-\xi^2} \sqrt{1-k^2\xi^2}} d\xi,$$

$$k = \nu = \frac{1 + \cos \lambda \cos \phi}{1 - \cos \lambda \cos \phi}, \quad z = \frac{\cos \lambda - 1}{\sin \lambda} \sqrt{k}.$$

It can be checked that for all $G(\eta_1, \eta_2)$ having bounded partial derivatives for $\eta_1 = 0$ and all η_2 we have

$$\nabla \pi_1(t, \lambda, \phi) \Big|_{(\lambda, \phi)=0} = 0.$$

For the sake of simplicity we take $\pi_1 = 0$. Then the trajectory of the steady vortex can be obtained from (3.10). In particular, for $\Omega = 0$ (non-rotating case) and $l_0 = 0$ we get a stationary exact solution $(\mathbf{u}(\lambda, \phi), \pi_0(\lambda, \phi))$.

Since system (5.1), (3.6) has many solutions dependent on the choice of function Φ , one can construct a great variety of couples of "cyclonic" and "anticyclonic" steady vortices on the non-rotating sphere.

If $\Omega \neq 0$ and/or $\pi_1 \neq 0$, then the position of the respective vortex can be found from equation (3.10):

$$\ddot{\mathbf{X}}(t) + 2\Omega \sin(X_2(t)) L \dot{\mathbf{X}}(t) + c_0 \nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} = 0. \quad (5.8)$$

The discrepancy term for $A = 2\Omega \sin X_2(0)$ is

$$\begin{aligned} Q - F &= 2\Omega \left(\sin(\phi + X_2(t)) - \sin X_2(t) \right) L \dot{\mathbf{X}}(t) + \\ &2\Omega (\sin(\phi + X_2(t)) - \sin(X_2(0))) L \mathbf{u}(\mathbf{x}) + c_0 \left[\nabla \pi_1(t, \mathbf{x}) \Big|_{\mathbf{x}=0} - \nabla \pi_1(t, \mathbf{x}) \right], \end{aligned} \quad (5.9)$$

here $\mathbf{x} = (\lambda, \phi)$, $\mathbf{u} = (u, v)$. For $F = 0$ it can be considered as small for small t in a small neighborhood of the center of the theoretical vortex.

For $a = b = 0$ in (5.2) condition (3.16) is satisfied for $A = 2\Omega \sin \phi L$, the respective pressure field is

$$\pi_0(\lambda, \phi) = -\frac{1}{4} h (h + 2\Omega) \sin 2\phi + C.$$

Thus, we get an exact solution with $\pi_1 = 0$ for the rotating sphere, moving according to (5.8). But evidently it is not a vortex solution, it can be better interpreted as a zonal flow.

We do not dwell here on the computations made for the case of the spherical geometry. This is a difficult issue, we reserve it for our future research. Difference methods that can be useful here are discussed in [3], [36], [28].

6. Discussion. The vortex motion is intrinsic to the fluid, both compressible and incompressible. There is a huge literature, dedicated to the subject (e.g. [19],[33],[21],[22]), especial emphases on vortex of rotating fluid and geophysical applications was made in [32] and [13]. In particular, the earliest results concerning vortices in the compressible fluid were obtained by C.Chree (e.g.[9]), it is remarkable that the author always bears in mind the meteorological context. Recently the vortices in compressible fluid were extensively studied by Shivamoggi (e.g.[35]).

It is interesting that a very complicated systems of equations describing the motion of compressible fluid possesses global in time solutions of very simple form. For example, in the Euclidean space it is a solution with linear profile of velocity, known since Kirkhoff. Meanwhile the solution is not physically reasonable since the velocity and the respective pressure field rise unboundedly as the space variables go to infinity.

The present study was inspirited by a well known fact that the tropical cyclones (typhoons) are very stable atmospherical structures and moves as a "rigid body" in a background field of pressure. There were many attempts to use this fact for a description of the trajectory of the vortex. It is possible to apply a beautiful theory of point vortex (point singularity) at a sphere [4] to geophysical problems [15], [5], [6]. Further, there is a number of paper, where the problem on the typhoon trajectory was solved in assumption that the vortex is a square root singularity, e.g. [7], [10]. Nevertheless, the assumption on a singular vortex structure contradicts to the data of observations [34], which evidence the linear profile of the velocity field near the eye of typhoon.

We propose a method of computing the position of the center of vortex. The technique can be used for any Riemannian manifold, nevertheless our main interest is a sphere and a plane as a local flat approximation of the sphere near a point. We consider two types of approximations: the l -plane model and the β - plane model.

We solve a problem of finding the trajectory of a steady vortex moving with a bearing field of pressure. Namely, we consider a special kind of solution to the system describing the motion of compressible fluid on a rotating two-dimensional manifold.

Our method consists of several steps.

a) We use the fact that the vertical scale of the atmosphere is small compared with the horizontal one. This allows to average the whole atmosphere model over the height and to reduce it to two space dimensions. The vertical structure plays the crucial role on the phase of formation and maintenance of the stable vortex. The averaging procedure hides these vertical processes, this simplifies the model and helps to manage with the problem of the trajectory describing. For the sake of simplicity we dwell on the barotropic case.

b) Further, as is known from experiments, near the eye of cyclone the motion is axial-symmetric, the divergency vanishes and the tangential component of velocity rises linearly (see [34]). We put the origin at the center of the vortex having coordinates $(X_1(t), X_2(t))$ (here we follow [7]) and in the new coordinates (t, x_1, x_2) we perform a procedure that can be called an approximate separation of variables. Namely, we separate all terms into three parts: the first depends only on the new space variables (x_1, x_2) , the second depends only on t , and the third depends both on space and time variables however from some reasons can be considered as small. In this connection we use a notion of the δ -approximate solution of the reduced system of atmospheric motion in a specific subdomain of the manifold where the system is given.

c) Given a vortex structure we find the stable component of (re-normalized) pres-

sure corresponding to the vortex. Further from a linear PDE we find the bearing part of the pressure field. Then we get a second order ODE for the vortex center position $X(t)$ (in general case it includes the distance from the center as a small parameter). In the case of a sphere we find exact stationary solution of the planetary scale (non-rotating case) and study an approximate vortex solution in the rotative case.

We investigate the question on the difference between trajectory of non-localized vortex on the l - plane where the exact solution can be obtained and the trajectories of localized vortex on l - plane and β - plane that seems more realistic. We show that the difference is reasonably small.

We do not stop here on the case of the presence of (turbulent) viscosity in the model. This situation can be considered in our framework. The difference is in the form of the master equation, in other words, the problem is to find a stationary solution to the Navier-Stokes equation. The class of such solution is not so rich as in the non-viscous case, we refer for a comprehensive review of this question to [38].

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